

Iwahori Hecke algebras

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Tutorial 1

Notation: $G = \mathrm{SL}_2(\mathbb{F}_q)$, $B \subset G$ is the subgroup of upper triangular matrices.

1. Bruhat decomposition: Prove that there is a decomposition

$$G = B \sqcup BwB,$$

where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

2. Write an explicit basis of the Hecke algebra

$$\mathcal{H}(G, B) := \{f : G \rightarrow \mathbb{C} \mid f(bgb') = f(g) \text{ for all } g \in G \text{ and } b, b' \in B\}.$$

3. Prove that a complete set of left coset representatives of B in G is

$$\left\{ \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix} \mid \mu \in \mathbb{F}_q \right\} \cup \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

4. Use the convolution product to calculate relations among the basis vectors explicitly.
5. Do the exercises above with G replaced with $\mathrm{SL}_n(\mathbb{F}_q)$. The disjoint union in exercise 1. will be over the Weyl group W of G .
6. Find an I -fixed vector in St .